MATHEMATICS

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AIM POINT
MATHEMATICS
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APPLICATION OF DERIVATIVES

& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ Derivative as the Rate of Change

If a variable quantity y is some function of ime t ie, y = f(t), then small change in time Δt have a corresponding change Δy in y. Thus, avrage rate of change = $\frac{\Delta y}{\Delta t}$.

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to t at the instant t.

ie,
$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

Motion in a Straight Line

If x and v denotes the displacemet and velocity of a particle at any instant t, then velocity is given by

$$v = \frac{dx}{dt}$$

and
$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d^2x}{dt^2}$$

Where a is acceleration of particle. If the sign of acceleration is opposite to that of velocity, then the acceleration is called retardation which means decrease in magnitude of the velocity.

✷ Increasing Funtion

These functions are of two types

1. Strictly increasing function
2. Non-decreasing function

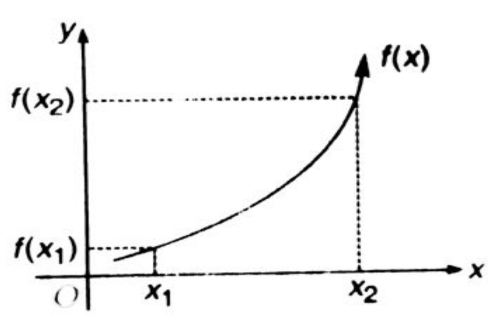
1. Strictly increasing function

A function f(x) is known as strictly increasing function in its domain, if

$$x_1 < x_2$$
$$\Rightarrow \quad f(x_1) < f(x_2)$$

Therefore, for the smaller input, we have smaller output and for higher value of input we have higher output.

Graphically it can be expressed as, shown in the adjoing figure.



Here, $x_1 < x_2 = \quad f(x_1) < f(x_2)$

Thus, f(x) is strictly increasig function.

In the graph,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

As $x_1 < x_2 \Rightarrow \quad f(x_1) < f(x_2)$

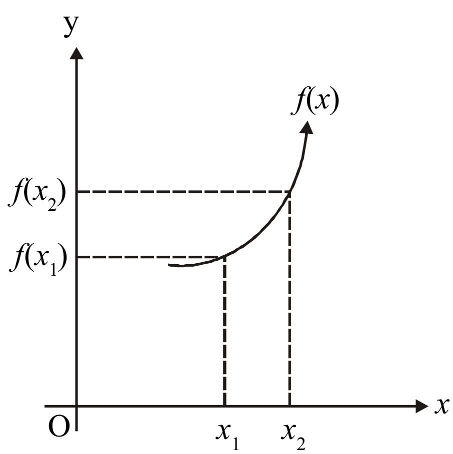
Thus, $f(x) < f(x + \text{h})$

$$f'(x) = \lim_{h \to 0} \frac{positive}{positive}$$

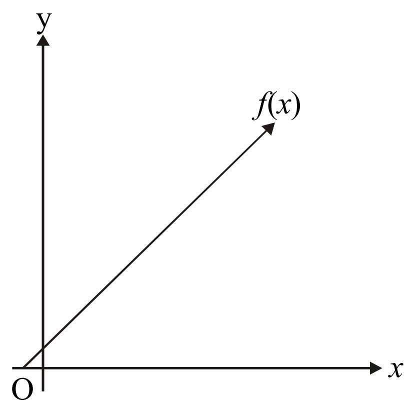
ie, $f'(x) > 0$

Thus, $f(x)$ will be strictly increasing if $f'(x) > 0 \ \forall \ x \in$ domain.

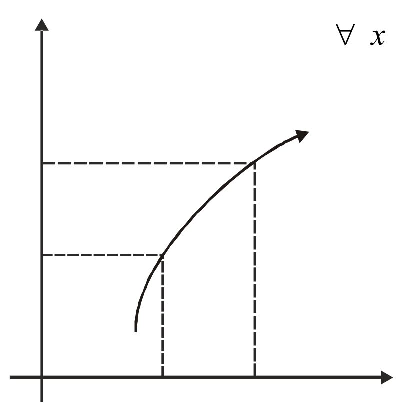
Classificatin of Strictly Increasing Function



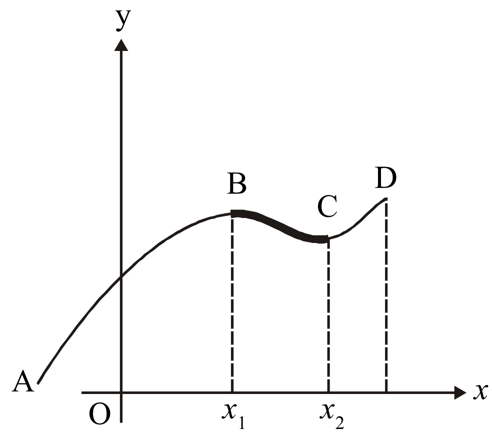
Concave up
when $f'(x) > 0$
and $f''(x) > 0$,
$\forall \ x \in$ domain



Neither concave up nor concave down
when $f'(x) > 0$
and $f''(x) = 0$,
$\forall \ x \in$ domain



Concave down
when $f'(x) > 0$
and $f''(x) < 0$,
$\forall \ x \in$ domain



2. Non-decreasing Function

A function $f(x)$ is said to be non-decreasing, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) \leq f(x_2)$

As shown in adjoint figure.

For AB and CD portion, $x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2)$

and for BC, $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-decreasing function (or increasing function),

if $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2)$

Obviously, for this $f'(x) > 0$, where equality holds for horizontal path of the graph ie, in the interval of BC.

✷ **Decreasing Function**

These function are also of two types

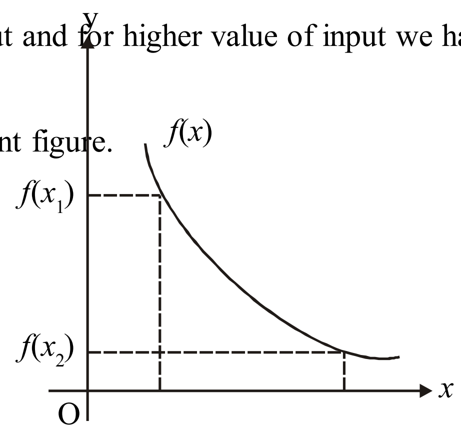
1. Strictly decreasing function
2. Non-increasing function

1. Strictly decreasing function

A function $f(x)$ is known as strictly decreasing function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Therefore, for the smaller input we have higher output and for higher value of input we have smaller output.

Graphically it can be expressed as shown in the adjoint figure.



Here, $\quad x_1 < x_2$

$\Rightarrow f(x_1) > f(x_2)$ thus, $f(x)$ is strictly decreasing.

In graph,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

As $x_1 < x_2 = f(x_1) > f(x_2)$

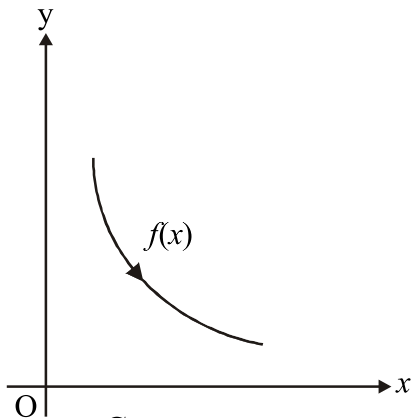
Thus, $\quad f(x + \text{h}) < f(x)$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{-ve}{+ve}$$

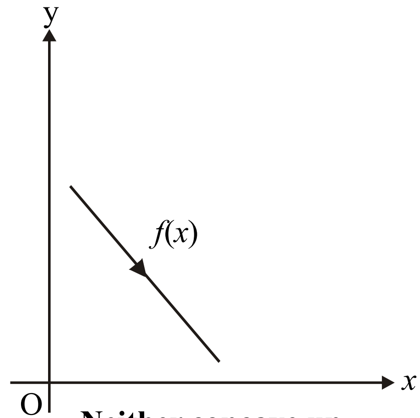
ie, $\quad f'(x) < 0$

Thus, $f(x)$ will be strictly decreasing, if $f'(x) < 0 \;\; \forall \;\; x \in$ domain.

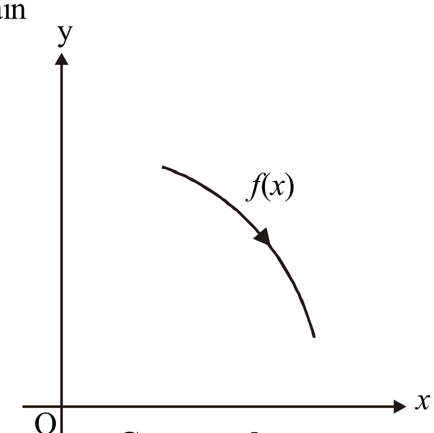
Classificatin of Strictly Decreasing Function



Concave up
when $f'(x) < 0$
and $f''(x) > 0$,
$\forall\ x \in$ domain

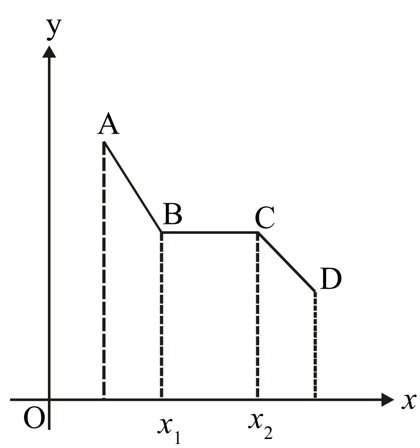


Neither concave up nor concave down
when $f'(x) < 0$
and $f''(x) = 0$,
$\forall\ x \in$ domain



Concave down
when $f'(x) < 0$
and $f''(x) < 0$,
$\forall\ x \in$ domain

2. Non-increasing function



A function $f(x)$ is said to be non-increasing, if for $x_1 < x_2$.

$\Rightarrow \quad f(x_1) \geq f(x_2)$

As shown in adjoint figure.

For AB and CD portion, $x_1 < x_2$

$\Rightarrow \quad f(x_1) > f(x_2)$

and for BC, $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) = f(x_2)$

Hence, as a whole we can say that for non-increasing function (or decreasing function),

if $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) \geq f(x_2)$

Obviously, for this $f'(x) \leq 0$, where equality holds for horizontal path of the graph ie, in the interval of BC.

✱ Monotonic Function

A function f is said to be monotonic or monotone in an interval I. If it either increasing or decreasing in the interval I. $f(x) = \text{In } x$, $f(x) = 2^x$, $f(x) = -2x + 3$ are monotonic function. $f(x) = x^2$ is monotonic in $(-\infty, 0)$ or $(0, \infty)$ but is not monotonic in R.

Properties of Monotonic Function

(i) If $f(x)$ is continuous on [a, b] such that $f'(c) < 0$ ($f'(c) < 0$) for each $c \in (a, b)$, then $f(x)$ is monotonically (or strictly) decreasing function on [a, b].

(ii) If $f(x)$ is continuous on [a, b] such that $f'(c) > 0$ [$f'(c) > 0$] for each $c \in (a, b)$, then $f(x)$ is monotonically (sbtrictly) increasing function on [a, b].

(iii) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) function on [a, b], then $gof(x)$ is a monotonically (or strictly) increasing function on [a, b].

(iv) If one of the two function $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decriasing on [a,b].

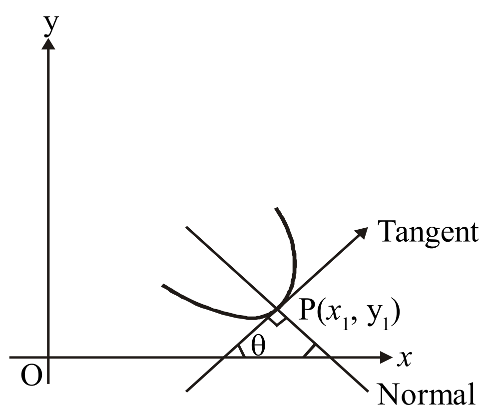
(v) If $f(x)$ is strictly increasing function on an interval [a, b], then f^{-1} exists and it is also strictly increasing function.

(vi) If $f(x)$ is strictly increasing function on an interval [a, b] such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

✱ Tangent and Normal

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_2)$ be the point on it.

Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of tangent to the curve $y = f(x)$ at a point P.



ie, $$\left(\frac{dy}{dx}\right)_P = \tan\theta$$

$$= \text{Slope of tangent at P}$$

where, t is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of x-axis as shown in the figure.

Equation of Tangent

The equation of tangent to any curve ta the point $P(x_1, y_1)$ is $(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$.

Slope of Normal

We know that normal to the curve at P(x_1, y_1) is a line perpendicular to tangent at P(x_1, y_1) and passing through P.

$\therefore$ Slope of the normal at P = $-\dfrac{1}{\text{slope of the tangent at P}}$

$\Rightarrow$ Slope of the normal at P(x_1, y_1) = $-\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}$

$\Rightarrow$ Slope of the normal at P(x_1, y_1) = $-\left(\dfrac{dx}{dy}\right)_{(x_1,y_1)}$

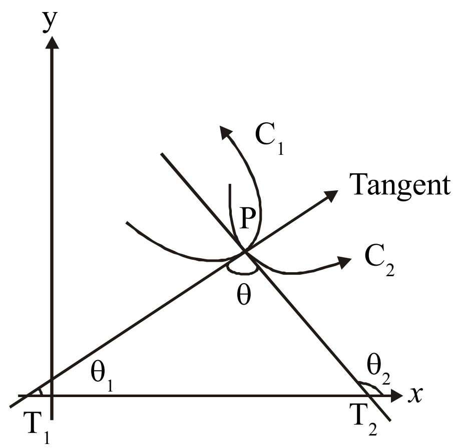
Equation of Normal

The equation of normal to any curve at the point P(x_1,y_1) is

$$(y-y_1)=-\left(\frac{1}{dy/dx}\right)_{(x_1,y_1)}(x-x_1)$$

✶ **<u>Angle of Intersection of Two Curves</u>**

The angle of intersection of two curves is the angle subtended between the tangents at their point of intersection.



Let C_1 and C_2 be two curves having equation $y = f(x)$ and $y = g(x)$ respectively.

Let θ be the angle between the two tangents PT_1 and PT_2 and θ_1 and θ_2 are the angles made by tangents with the positive direction of x-axis in anti-clockwise sense.

Then, $\quad m_1 = \tan\theta_1 = \left(\dfrac{dy}{dx}\right)_{C_1}$

and $\quad m_2 = \tan\theta_2 = \left(\dfrac{dy}{dx}\right)_{C_2}$

From the figure it follows,

$$\theta = \theta_1 - \theta_2$$

$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2 \tan\theta_1}$$

$\Rightarrow$ $$\tan\theta = \left|\frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1+\left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}\right| = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$$

Orthogonal Curves

If the angle of intersection of two curves is right angle, then two curves are said to be orthogonal curves.

If the curves are orthogonal, then $\theta = \frac{\pi}{2}$

$\Rightarrow$ $$1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

$\Rightarrow$ $$m_1 m_2 = -1$$

✷ **<u>Length of Tangent, Normal, Subtangent and Subnormal</u>**

Length of Tangent

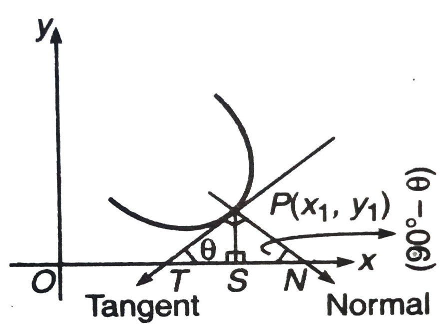
The length of the segment PT of the tangent between the point the intersection and x-axis is called the

$\therefore$ Length of tangent $= \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$

Length of Normal

The length o the segment PN of the normal intercepted between the point on the curve and x-axis, is called the length of normal.

$\therefore$ Length of normal $= \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$



Length of Subtangent

The projection of the segment PT along x-axis (ST) is called the subtangent.

$$\therefore \quad \text{Length of Subtangent} = \left|\frac{y}{dy/dx}\right|$$

Length of Subnormal

The projection of the segment PN along x-axis (SN) is called the subnormal.

$$\therefore \quad \text{Length of normal} = \left|y\left(\frac{dy}{dx}\right)\right|$$

✱ **Critical Points**

It is a collection of Points for which,

1. $f(x)$ does not exists.
2. $f'(x)$ does not exists or
3. $f'(x) = 0$

All the values of x obtained from above conditions are said to be critical points.

It should be noted that critical points are the interior points of an interval.

✱ **Maxima and Minima**

Let y = $f(x)$ be a function defined at $x =$ a and also in the neihbourhood of x = a. Then, $f(x)$ is said to have a maximum of local maxmum at x = a, if the value of the function at x = a is greater than the value of the function at the neighbourhood points of x = a.

ie, $f(a) > f(a + h)$ and $f(a) > f(a - h)$, where $h > 0$.

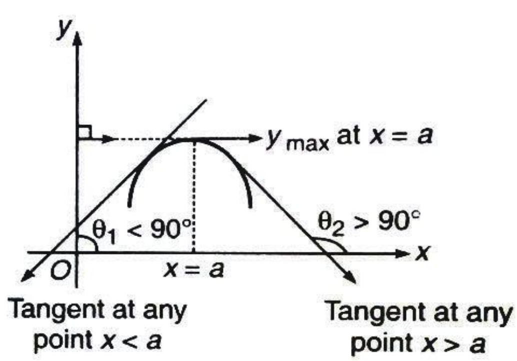
Similarly, $f(x)$ is said to have minimum or local minimum at x = a, if the value of the function at x = a is less than the value of the function at the neighbourhood points of x = a. ie, $f(a) < f(a + h)$ and $f(a) < f(a - h)$, where $h > 0$.

✱ **Maxima and Minima**

1. First Derivtive Test

The following test applies to a continuous function in order to get the extrema.

As we know that the function attains maximum, when it has taken its maximum value and attains minimum, when it has taken its minimum value which could be shown as,



At a Critical Point $x = x_0$

(i) When $f(x)$ attains maximum at x = a.

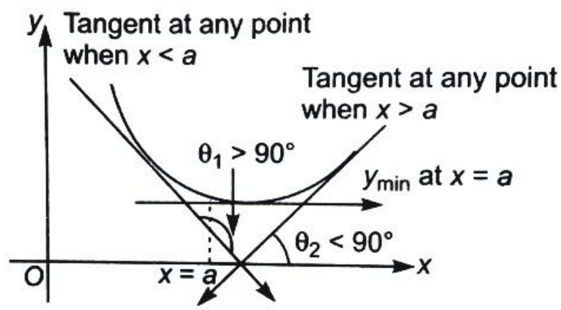
ie, from the above graph.

$$\begin{cases} \text{for } x < a,\ \theta_1 < 90^o = \tan\theta_1 > 0 \text{ or increasing for } x < a. \\ \text{for } x = a,\ \tan\theta = 0, \text{ or neither increasing nor decreasing.} \\ \text{for } x > a,\ \theta_2 > 90^o = \tan\theta_2 < 0 \text{ or decreasing for } x < a. \end{cases}$$

Thus, we can say, $f(x)$ is maximum at some point (x = a)

$$\Rightarrow \begin{cases} f(x) \text{ is increasing for } x < a \\ f(x) \text{ is decreasing for } x > a \end{cases}$$

(ii) When $f(x)$ attains minimum at x = a



ie, from the above graph.

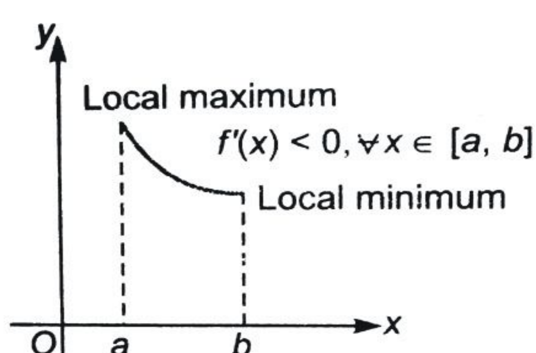
$$\begin{cases} \text{for } x < a,\ \theta_1 < 90^o = \tan\theta_1 > 0 \text{ or decreasing for } x < a. \\ \text{for } x = a,\ \tan\theta = 0, \text{ or neither increasing nor decreasing.} \\ \text{for } x > a,\ \theta_2 > 90^o = \tan\theta_2 < 0 \text{ or increasing for } x < a. \end{cases}$$

Thus, we can say, $f(x)$ is maximum at some point (x = a)

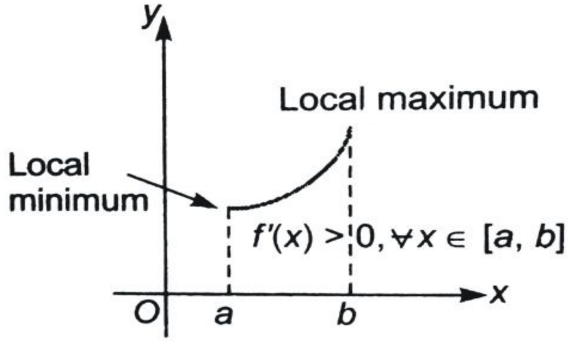
$$\Rightarrow \begin{cases} f(x) \text{ is decreasing for } x < a \\ f(x) \text{ is increasing for } x > a \end{cases}$$

At a Left End Point a and Right End Point b in [a, b]

Let $f(x)$ be defined on [a, b].



If $f'(x) < 0$, for $x > a$, then $f(x)$ has local maximum at $x = a$ and local minimum at $x = b$.



Again, if $f'(x) > 0$, for $x > a$, then $f(x)$ has local minimum at $x = a$ and local maximum at $x = b$.

2. Second Derivative Test

First we find the roots of $f'(x) = 0$. Suppose $x = a$ is one of the roots of $f'(x) = 0$.

Now, find $f''(x)$ at $x = a$.

(a) If $f''(a)$ = negative; then $f(x)$ is maximum at $x = a$.

(b) If $f''(b)$ = positive; then $f(x)$ is minimum at $x = a$.

(c) If $f''(c)$ = zero; then we find $f'''(x)$ at $x = a$.

If $f'''(a) = 0$, then $f(x)$ has neither maximum nor minimum (inflexion point) at $x = a$.

But, if $f'''(a) = 0$, then find $f^{iv}(a)$;

If $f^{iv}(a)$ = positive, then $f(x)$ is minimum at $x = a$.

If $f^{iv}(a)$ = negative, then $f(x)$ is maximum at $x = a$.

and so on, process is repeated till point is discussed.

✱ Absolute Maxima and Absolute Minima

Absolute (or Global) maxima or minima of $f(x)$ in [a, b] is basically the greatest of least value of $f(x)$ in [a, b] and it will always occur either at the critical points of $f(x)$ with in [a, b] or at the end points of the interval.

Absolute Maxima of Minima in [a, b]

Step I Find out all the critical points of f(x) in (a, b). Let $c_1, c_2, \ldots\ldots, c_n$ be the different critical points.

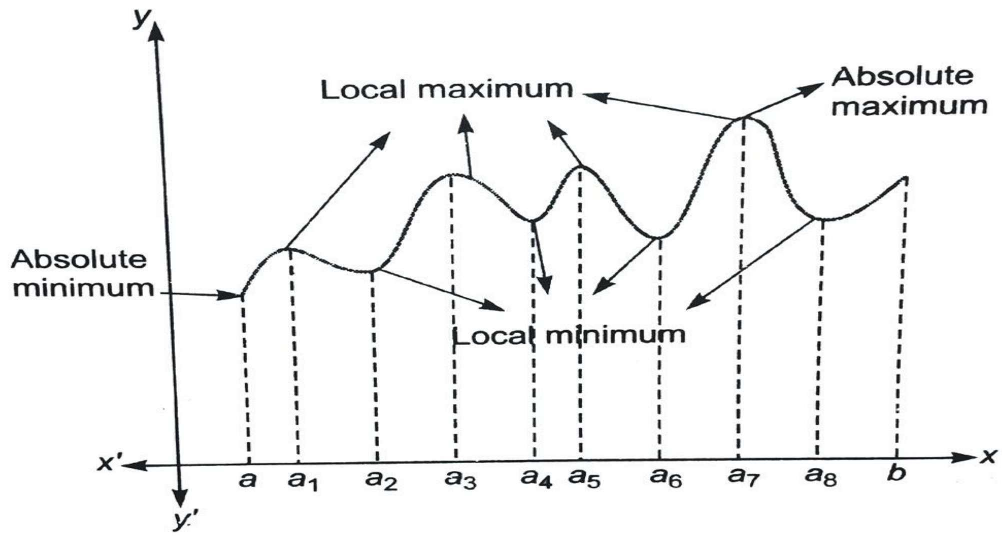
Step II Find the value of the function at there critical points and also at the end points of the domain. Let the value are $f(c_1), f(c_2), \ldots\ldots f(c_n)$.

Step III Find $M_1 = \max \{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$

and $M_2 = \min \{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$

Now, M_1 is the maximum value of $f(x)$ in [a, b], so M_2 is absolute maximum and M_2 is the min value of $f(x)$ in [a, b], so M_2 is absolute minimum.

Let $y = f(x)$ be the function defined on [a, b] in the graph, then



(i) $f(x)$ has local maximum values at $x = a_1, a_3, a_5, a_7$

(ii) $f(x)$ has local minimum values at

$$x = a_2, a_4, a_6, a_8$$

(iii) The absolute maximum value of the function is $f(a_7)$ and absolute minimum value is $f(a)$.

Absolute Maxima of Minima in [a, b]

To find the absolute maxima and minima in (a, b) step I and step II are same. Now,

Step III Find $M_1 = \max\{f(c_1), f(c_2), \ldots\ldots\ldots\ldots f(c_n)\}$

$M_2 = \min\{f(c_1), f(c_2), \ldots\ldots\ldots\ldots f(c_n)\}$

Now, if $\lim\limits_{x\to a^+ \text{ or } x\to b^-} f(x) > M_1$ or $\lim\limits_{x\to a^+ \text{ or } x\to b^-} f(x) > M_2$, then

$f(x)$ would not have absolute maximum of absolute minimum in (a, b)

and if $$\lim_{x\to a^+ \text{ or } x\to b^-} f(x) < M_1$$

and $$\lim_{x\to a^+ \text{ or } x\to b^-} f(x) > M_2$$

then M_1 and M_2 would respectively be the absolute maximum and absolute minimum of $f(x)$ in (a, b).

✶ **Rolle's Theorem**

If a function $f(x)$.

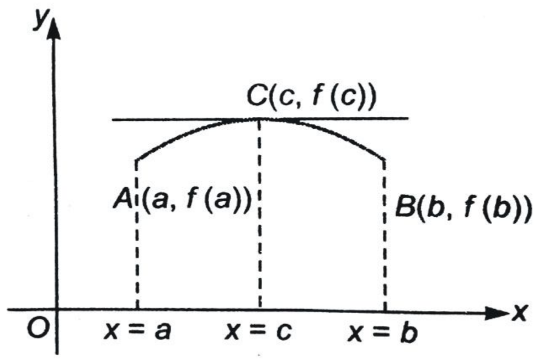
1. is continuous in the closed interval [a, b]
2. is differentiable in an open interval (a, b) ie, differentiable at each point in the open inverval (a, b).
3. and $f(a) = f(b)$

Then, there will be at least one point c in the interval (a, b) such that $f'(c) = 0$.

Geometrical Interpretation of Rolle' Theorem

Let $f(x)$ a real valued function on [a, b] such that curve $y = f(x)$ is continuous curve between points A(a, f(a)) and B(b, f(b)) and it is possible to draw unique tangent at every point on the curve between point A and

B. Also, the ordinates at the end point of the interval [a, b] are equal. Then, there exists at least one point (c, f(c)) between A and B on the curve where tangent is parallel to x-axis.



✷ Lagrange's Mean Value Theorem

If a function $f(x)$ is.

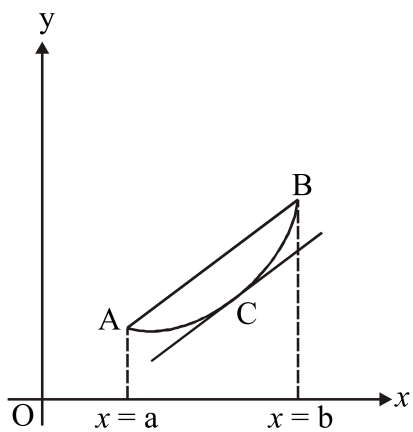
1. continuous in the closed interval [a, b]
2. differentiable in an open interval (a, b)

Then, there exist at least on point c, where a < c < b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrial Interpretation of Lagrane's Mean Value Theorem.

Let $f(x)$ be a real valued function on [a, b] such that the curve y = $f(x)$ is continuous curve between points A(a, f(a)) and possible to draw a unique tangent. Then, there exist a point on the curve such that the tangent at this point is parallel to the chord joining the end points of the curve.



Note :

- The differential coefficient of y with respect to x ie, $\frac{dy}{dx}$ is nothing but the rate of increase of y relative to x.
- If tangent is parallet to x-axis then

$$\theta = 0^o$$

$$\Rightarrow \quad \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

- If tangent is perpendicular to x-axis (for parallel to y-axis), then

$$\theta = 90^o = \tan\theta = \infty \quad \text{or} \quad \cot\theta = 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

- If normal is parallel to x-axis.

$$\Rightarrow \quad -\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

- If normal is perpenducalar to x-axis (or parallel to y-axis).

$$\Rightarrow \quad -\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

- If a function is strictly increasing in [a, b], then $\begin{cases} f(\text{a}) \text{ is local minimum} \\ f(\text{b}) \text{ is local maximum} \end{cases}$
- If a function is strictly decreasing in [a, b], then $\begin{cases} f(\text{a}) \text{ is local maximum} \\ f(\text{b}) \text{ is local minimum} \end{cases}$
- It must be remembered that this method this method is not applicable to those critical points, where $f'(x)$ remains undefined.
- Between two local maximum values, there is a local minimum value and vice-versa.
- A local minimum value may be greater than a local maximum value. In the above graph, local minimum at a_6 is greater than local maximum at a_1.